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Type-II hidden symmetries of the linear 2D and 3D wave equations

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Abstract

Type-II hidden symmetries of the linear, two-dimensional and three-dimensional wave equations are analysed. These hidden symmetries are Lie point symmetries that appear in addition to the inherited point symmetries when the number of independent and dependent variables of a partial differential equation is reduced by a Lie point symmetry. The provenance of these hidden symmetries of partial differential equations is identified to be the same as found recently for some nonlinear partial differential equations. The appearance of Type-II hidden symmetries depends not only on the Lie symmetries used but on the order in which the symmetries are applied. The presence of Type-II hidden symmetries of partial differential equations complicates the prediction of symmetry reductions based on the Lie algebra associated with the original Lie point symmetries.

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1. Introduction

Many scientific and engineering problems are formulated in terms of partial differential equations (PDEs). Analyses of the symmetries of partial differential equations have produced many useful analytical and numerical solutions. The symmetries may be the classical Lie symmetries or generalized symmetries [1–8]. The ultimate aim of the symmetry analysis is to discover solutions of the PDEs that obey the boundary and initial conditions.

The classical Lie symmetries of PDEs are now mostly calculated symbolically by computer programs. The number of dependent and independent variables of a PDE can be reduced by one if a Lie symmetry is used to define new variables. The resultant reduced differential equation loses the symmetry used to reduce the number of variables and it may lose other Lie symmetries depending on the structure of the associated Lie algebra [9]. For many PDEs the Lie symmetries at each subsequent reduction are the inherited Lie symmetries from the original PDE. There are exceptions to this rule although this fact [5, 10, 11] may not be widely known. We recently presented an explanation for the appearance of Lie symmetries of a reduced differential equation that were not inherited from the preceding nonlinear PDE [12]. These symmetries are Type-II hidden symmetries but differ in origin from Type-II hidden symmetries of ordinary differential equations (ODEs) [13–25]. The Type-II hidden symmetries of ODEs were found by several approaches including solvable structures that involve differential forms [21]. The differential forms had been used previously to analyse symmetries of PDEs and the prolongation Lie algebra of soliton equations [26, 27]. The significance of these Type-II hidden symmetries is that there may be more symmetries in the subsequent reduced differential equations than can be predicted from the Lie algebra of the original PDE. The general premise of this paper is that increased understanding of Type-II hidden symmetries as a part of Lie symmetries is a useful endeavour and may lead to improvements in the solution of differential equations. The research was initially motivated by several questions: (1) the occurrence of Type-II hidden symmetries of PDEs, (2) the provenance of Type-II hidden symmetries of PDEs and (3) the prediction of Type-II hidden symmetries from the original PDE. New topics have subsequently have arisen: (4) the possibility of missing Type-II hidden symmetries if only inherited symmetries are used in the reduction of PDEs and (5) new techniques to find all Lie point symmetries in reduced PDEs. The last is important if computer programs do not compute all the symmetries of the PDEs. Topic (3) is not discussed in this paper but it depends on topics (1) and (2).

The Type-II hidden symmetries of two common linear PDEs, the two-dimensional and three-dimensional wave equations, and their descendants are presented here. The Lie symmetries of the linear wave equations in rectangular, Cartesian coordinates are well known. On the other hand the existence of the Type-II hidden symmetries in the reductions of these wave equations does not appear to have been reported with one exception [5] until recently [28]. The wave equations have extensive physical applications so that any additional exact solutions could be of interest. The analysis of the Type-II hidden symmetries of the wave equations is not exhaustive; for example, a travelling wave solution of the linear three-dimensional wave equation reduces it to a linear two-dimensional wave equation.

The provenance of Type-II hidden symmetries of the descendant (reduced) differential equations of two nonlinear PDEs was found to be inherited symmetries of one or more other PDEs that reduced to the same descendant differential equations. It was clear that the Type-II hidden symmetries of these PDEs were not found from nonlocal or contact symmetries as was true for ODEs. This holds since only variable transformations of the PDEs are used. The provenance of the Type-II hidden symmetries of the linear wave equations is the same as that identified for the nonlinear PDEs analysed recently although the provenance is more subtle. These hidden symmetries of PDEs are not the Type-I hidden symmetries that disappear when the number of the variables is reduced [29], nor are they nonlocal hidden symmetries [30] or Q-conditional symmetries of PDEs that can be hidden [31].

2. Type-II hidden symmetries of the linear, three-dimensional wave equation

The linear, three-dimensional wave equation is

$$u_{xx} + u_{yy} + u_{zz} - u_{tt} = 0 \tag{1}$$

where u is the wave function, x, y and z are the spatial coordinates and t is the time normalized by the wave speed. The Lie point symmetries are identified by the group generators

$$U_i = \xi_x^i \frac{\partial}{\partial x} + \xi_y^i \frac{\partial}{\partial y} + \xi_z^i \frac{\partial}{\partial z} + \xi_t^i \frac{\partial}{\partial t} + \eta_u^i \frac{\partial}{\partial u}, \qquad i = 1, \ldots, 16,$$
$$U_\infty = F_u(x, y, z, t)\frac{\partial}{\partial u}. \tag{2}$$

The infinitesimals ξ_x^i, ξ_y^i, ξ_z^i, ξ_t^i and η_u^i are functions of x, y, z, t and u. There are 16 group generators and U_∞ where $F_u(x, y, z, t)$ is a solution of the linear PDE (1). The Lie symmetries in (2) are known [7], were checked by the computer program LIE [32], and are listed in (A.1) in the appendix. Stephani [5] gave only the symmetries used in reduction.

We next follow the successive reductions of the numbers of variables of (1) as performed by Stephani [5] until an ODE is reached. The symmetries used (in order) are scaling, rotation in the x–y plane and Lorentz transformation or pseudo rotation. The general rule is that the number of independent and dependent variables is reduced by one when new variables are defined with one Lie point symmetry. The inherited Lie point symmetries are identified from the commutator of the symmetry used to reduce the number of variables U_α and another group generator U_j:

$$[U_\alpha, U_j] = C_k^{\alpha j} U_k. \tag{3}$$

If $C_k^{\alpha j} = 0$ or $\alpha = k$, the symmetry of U_j is inherited. Otherwise the symmetry is lost.

The wave equation is reduced by the scaling symmetry to

$$(1-\bar{x}^2)w_{\bar{x}\bar{x}} + (1-\bar{y}^2)w_{\bar{y}\bar{y}} + (1-\bar{z}^2)w_{\bar{z}\bar{z}} - 2\bar{x}\bar{y}w_{\bar{x}\bar{y}} - 2\bar{x}\bar{z}w_{\bar{x}\bar{z}}$$
$$- 2\bar{y}\bar{z}w_{\bar{y}\bar{z}} - 2\bar{x}w_{\bar{x}} - 2\bar{y}w_{\bar{y}} - 2\bar{z}w_{\bar{z}} = 0 \tag{4}$$

with $\bar{x} = x/t$, $\bar{y} = y/t$, $\bar{z} = z/t$ and $w = u$. The inherited Lie point symmetries of (4) are

$$V_i = \xi_{\bar{x}}^i \frac{\partial}{\partial \bar{x}} + \xi_{\bar{y}}^i \frac{\partial}{\partial \bar{y}} + \xi_{\bar{z}}^i \frac{\partial}{\partial \bar{z}} + \eta_w^i \frac{\partial}{\partial w}, \qquad i = 1, \ldots, 7,$$
$$V_\infty = F_w(\bar{x}, \bar{y}, \bar{z})\frac{\partial}{\partial w}. \tag{5}$$

The infinitesimals in (5) are given in (A.2) in the appendix and $F_w(\bar{x}, \bar{y}, \bar{z})$ is a solution of (4). The symmetries in (A.1) inherited in (A.2) are indicated in the appendix. The determining equations for the inherited Lie point symmetries of (4) were computed by Maple and solved using Maple after the symmetries were not found by the computer program LIE [32] and the computation by hand seemed formidable. The inherited Lie symmetries were shown to be the only Lie symmetries of (4).

The second reduced PDE uses the symmetry of the rotation in the original x–y plane that is inherited as V_4 in (A.2) in the variables of (4). The resultant PDE is

$$4v(1-v)w_{vv} - 4vsw_{vs} + (1-s^2)w_{ss} + (4-6v)w_v - 2sw_s = 0 \tag{6}$$

with $v = \bar{x}^2 + \bar{y}^2$, $s = \bar{z}$. The three Lie point symmetries of (6) are

$$X_1 = 2sv\frac{\partial}{\partial v} + (s^2-1)\frac{\partial}{\partial s}, \qquad X_2 = w\frac{\partial}{\partial w}, \qquad X_\infty = F_w(v, s)\frac{\partial}{\partial w} \tag{7}$$

where $F_w(v, s)$ is a solution of (6). Again the determining equations for the inherited Lie point symmetries of (6) were computed by Maple and solved using Maple as LIE did not find them and are the only Lie symmetries of (6).

The final reduction is to an ODE and is by the Lie point symmetry X_1. The ODE is

$$\sigma w_{\sigma\sigma} + w_\sigma = 0, \tag{8}$$

with $\sigma = \frac{v}{1-s^2}$. The ODE has two inherited symmetries and six Type-II hidden symmetries. The symmetries can be determined by the computer program LIE or by a hand calculation.

However, the simplest way to find the symmetries is to note that if we transform the independent variable to $\gamma = \ln \sigma$, then the ODE becomes

$$w_{\gamma\gamma} = 0. \tag{9}$$

This ODE is known to have the maximal number of Lie symmetries for a second-order ODE which is eight Lie point symmetries. The eight Lie symmetries of (8) are represented by

$$Y_j = \xi_\sigma^j \frac{\partial}{\partial \sigma} + \eta_w^j \frac{\partial}{\partial w}, \qquad j = 1, \ldots, 8 \tag{10}$$

and infinitesimals ξ_σ^j and η_w^j are given in (A.3) in the appendix. Stephani [5] identified only one hidden symmetry, $Y_3 = \sigma \frac{\partial}{\partial \sigma}$.

Type-II hidden symmetries of (8) are inherited symmetries of other PDEs than (6) that reduce to (8). If we restrict ourselves to the reduction of PDEs by the symmetry X_1, another PDE that reduces to (8) is

$$v^2 w_{vv} + v w_v = 0 \tag{11}$$

where $w(v, s)$. The determination of (11) is subtle even though its reduction to (8) is simple. This PDE can be guessed from (8) but was originally found by reverse transformations (see [12] for details but we review the procedure here). In the reverse transformations a group generator X_β that is a function of (v, s) is assumed to reduce to a hidden symmetry, here represented by Y_3. Then X_1 and X_β obey

$$[X_1, X_\beta] = C_1^{1\beta} X_1. \tag{12}$$

In this case we calculate X_β to be

$$X_\beta = \xi_s \frac{\partial}{\partial s} + \xi_v \frac{\partial}{\partial v} \tag{13}$$

where $\xi_s = C_1^{1\beta}(1 - s^2)\tanh^{-1} s, \xi_v = v\left(1 - \frac{2s\xi_s}{1-s^2}\right)$. The expression for ξ_s can be more general but a more general form for (11) has not been found. Two invariants from X_β are added together to form (11). Other invariants are calculated but they contain factors of $\tanh^{-1} s$. These factors do not disappear upon reduction of the invariants by the symmetry of X_1. Since the two invariants in (11) are the most general invariants for one hidden symmetry, other invariants from the remaining hidden symmetries cannot be included in (11). There may be other ODEs that reduce to (8) if another symmetry is used in the reduction but these have not been identified.

3. Type-II hidden symmetries of the two-dimensional wave equation

The linear two-dimensional wave equation is

$$u_{xx} + u_{yy} - u_{tt} = 0. \tag{14}$$

The first set of reductions of this PDE to an ODE is done by similar symmetries to those used to reduce the linear three-dimensional wave equation. The scaling symmetry and then the rotation symmetry in the x–y plane complete the reduction to one ODE. Reductions to other ODEs are also discussed. The symmetries of (14) are well known [7] and were checked by LIE.

However, the reduced PDE is more easily analysed in circular spatial coordinates. Therefore, the two-dimensional wave equation is rewritten as

$$r^2 w_{rr} + r w_r + w_{\theta\theta} - r^2 w_{tt} = 0 \tag{15}$$

where $r = (x^2 + y^2)^{1/2}$, $\tan\theta = y/x$ and $w = u$. The Lie point symmetries of the two-dimensional wave equation are

$$U_i = \xi_r^i \frac{\partial}{\partial r} + \xi_\theta^i \frac{\partial}{\partial \theta} + \xi_t^i \frac{\partial}{\partial t} + \eta_u^i \frac{\partial}{\partial u}, \qquad i = 1, \ldots, 11, \qquad U_\infty = F_u(r, \theta, t)\frac{\partial}{\partial u}. \tag{16}$$

Here the infinitesimals are given in (A.4) in the appendix and $F_u(r, \theta, t)$ is a solution of (15).

The reduced PDE found by the scaling symmetry U_5 in (A.4) is

$$R^2(1 - R^2)w_{RR} + R(1 - 2R^2)w_R + w_{\theta\theta} = 0 \tag{17}$$

with $R = r/t$, $u = w$. The symmetries of the reduced PDE (17) were determined in several ways. The reverse transformation from a symmetry of the reduced ODE revealed one Type-II hidden symmetry and two others were computed from the commutators with all the inherited symmetries. Next a hand calculation was performed with separation of variables used in the computation of the infinitesimals. The symmetries were most easily found by a coordinate transformation to Laplace's equation since the symmetries of that equation are known. If we let $z = \ln\left(\frac{R}{1+\sqrt{1-R^2}}\right)$, then (17) is transformed to Laplace's equation

$$w_{zz} + w_{\theta\theta} = 0. \tag{18}$$

Finally the Lie symmetries were checked by finding and solving the determining equations by Maple. The symmetries of (17) are four inherited symmetries and three Type-II hidden symmetries:

$$\begin{aligned} V_i &= \xi_R^i \frac{\partial}{\partial R} + \xi_\theta^i \frac{\partial}{\partial \theta} + \eta_w^i \frac{\partial}{\partial w}, \qquad i = 1, \ldots, 7 \\ V_\infty &= F_w(R, \theta)\frac{\partial}{\partial w}. \end{aligned} \tag{19}$$

The infinitesimals are given in (A.5) in the appendix and $F_w(R, \theta)$ is a solution of (17).

The PDE (17) is reduced to an ODE by use of the rotation symmetry V_1 in (A.5) to

$$R^2(1 - R^2)w_{RR} + R(1 - 2R^2)w_R = 0. \tag{20}$$

This ODE is invariant under the maximal number of symmetries for a second-order ODE, eight, as it can be transferred to $w_{zz} = 0$. The symmetries are represented by

$$X_i = \xi_R^i \frac{\partial}{\partial R} + \eta_w^i \frac{\partial}{\partial w}, \qquad i = 1, \ldots, 8. \tag{21}$$

It has three inherited symmetries and five Type-II hidden symmetries where the infinitesimals are in (A.6) in the appendix. Interestingly, one symmetry X_7 was inherited from a Type-II hidden symmetry of the reduced PDE (17). Both reduced differential equations from the linear two-dimensional wave equation have Type-II hidden symmetries in contrast to the reduced differential equations of the linear three-dimensional wave equation in section 2 that had Type-II hidden symmetries only in the reduced ODE.

Another PDE that reduces to (17) by the scaling transformation U_5 in (A.4) is

$$r^2[1 - (r/t)^2]w_{rr} + r[1 - 2(r/t)^2]w_r + w_{\theta\theta} = 0. \tag{22}$$

This was determined by finding the invariants by a reverse transformation and also by an educated guess. A PDE that reduces to the ODE (20) is

$$R^2(1 - R^2)w_{RR} + R(1 - 2R^2)w_R = 0 \tag{23}$$

for $w(R, \theta)$.

The reduction of the two-dimensional wave equation can be done by the same Lie point symmetries in the reverse order. If (15) is first reduced by the rotational symmetry, no Type-II hidden symmetries are found in the reduced PDE as can be seen by computing the symmetries with LIE and by computing the inherited Lie point symmetries. The Type-II hidden symmetries appear in the subsequent reduced ODE, however.

If we further reduce the reduced PDE (17) of the two-dimensional equation by the symmetry of a Type-II hidden symmetry, V_5, in (A.5), then a solution that would not be predicted from the Lie point symmetries of (15) is found. This is

$$w = K_1 \ln\left(\frac{1 - R\cos\theta}{1 + R\cos\theta}\right) + K_2 \tag{24}$$

for the constants K_1 and K_2 and is a solution of the two-dimensional wave equation. The 'similarity' variable is $z = R\cos\theta$ and the reduced ODE found by the symmetry is

$$(1 - z^2)w_{zz} - 2zw_z = 0. \tag{25}$$

Another solution could be found for U_6 in (A.5). The crucial result is that a solution of the two-dimensional wave equation is found that does not arise from inherited Lie point symmetries of the two-dimensional wave equation.

4. Discussion

Type-II hidden symmetries of the three-dimensional and two-dimensional linear wave equations have been presented. The Type-II hidden symmetries arise when the number of variables, independent in our cases here, is reduced by the use of Lie point symmetries. The provenance of these hidden symmetries has been shown previously to be point symmetries inherited from other PDEs. For each of the wave equations we have identified one other PDE from which the Type-II hidden symmetries have been inherited. The other PDEs have been restricted to PDEs that reduce to the same reduced differential equation by the same Lie point symmetry. For the nonlinear PDEs previously analysed the Type-II hidden symmetries were inherited from several other PDEs but there only one Type-II hidden symmetry appeared. The wave equations have so many Type-II symmetries in the ODEs, at least, that it is difficult to find a common set of invariants for all the inheritable symmetries that become the Type-II hidden symmetries. The other PDE from which the hidden symmetries are inherited are very similar to the reduced differential equation. This may explain why the origin of Type-II hidden symmetries had not been identified for some linear PDEs at least.

Our work indicates that, as in the case of ODES, merely looking at the Lie algebra of symmetries of the equation under analysis is not sufficient. Lie symmetries of all intermediate equations obtained via reduction must be calculated to see if other routes to reduction (and hence new solutions) can be obtained. Unfortunately, at this stage, it is not possible to determine *a priori* if hidden symmetries of PDEs may appear in a reduction (or indeed, as in the case of the 3D wave equation, when they will occur). It would be of interest to see if any prediction can be made via a differential geometric approach as was done in the case of ODEs [21].

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Appendix

The symmetry infinitesimals of differential equations discussed in the main text are given below except for one that is a solution of a linear PDE. Symmetry infinitesimals of (1) are

$$\begin{aligned}
\xi_x &= a_1 - a_2 y - a_6 z + a_8 t + a_{11} x - a_{13} x t - a_{14} x z - a_{15} x y + a_{16}\left(-\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} - \frac{t^2}{2}\right),\\
\xi_y &= a_2 + a_5 x - a_7 z + a_9 t + a_{11} y - a_{13} y t - a_{14} y z + a_{15}\left(\frac{x^2}{2} - \frac{y^2}{2} + \frac{z^2}{2} - \frac{t^2}{2}\right) - a_{16} x y,\\
\xi_z &= a_3 + a_6 x + a_7 y + a_{10} t + a_{11} z - a_{13} z t + a_{14}\left(\frac{x^2}{2} + \frac{y^2}{2} - \frac{z^2}{2} - \frac{t^2}{2}\right) - a_{15} y z - a_{16} x z,\\
\xi_t &= a_4 + a_8 x + a_9 y + a_{10} z + a_{11} t - a_{13}\left(\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} + \frac{t^2}{2}\right) - a_{14} z t - a_{15} y t - a_{16} x t,\\
\eta &= a_{12} u + a_{13} u t + a_{14} u z + a_{15} u y + a_{16} u x.
\end{aligned} \tag{A.1}$$

Symmetry infinitesimals of the first reduced PDE (4) of the three-dimensional wave equation are

$$\begin{aligned}
\xi_{\bar{x}} &= b_1(\bar{x}^2 - 1) + b_2 \bar{x}\bar{y} + b_3 \bar{x}\bar{z} + b_4 \bar{y} - b_5 \bar{z}\\
\xi_{\bar{y}} &= b_1 \bar{x}\bar{y} + b_2(\bar{y}^2 - 1) + b_3 \bar{y}\bar{z} - b_4 \bar{x} - b_6 \bar{z}\\
\xi_{\bar{z}} &= b_1 \bar{x}\bar{z} + b_2 \bar{y}\bar{z} + b_3(\bar{z}^2 - 1) + b_5 \bar{x} + b_6 \bar{y}\\
\eta_w &= b_7 w.
\end{aligned} \tag{A.2}$$

The symmetry generators are labelled with the subscript of the expansion coefficient of the infinitesimals. The group generators from (A.1) are inherited as follows:

$$\begin{array}{llll}
U_5 \to V_4, & U_6 \to V_5, & U_7 \to V_6, & U_8 \to V_1,\\
U_9 \to V_2, & U_{10} \to V_3, & U_{12} \to V_7, & U_\infty \to V_\infty.
\end{array}$$

Symmetry infinitesimals of the ODE (8) reduced from the second reduced PDE of the three-dimensional wave equation are

$$\begin{aligned}
\xi_\sigma &= d_3\sigma + d_4\sigma \ln\sigma + d_6\sigma(\ln\sigma)^2 + d_7\sigma w + d_8\sigma w \ln\sigma\\
\eta_w &= d_1 + d_2 \ln\sigma + d_5 w + d_6 w \ln\sigma + d_8 w^2.
\end{aligned} \tag{A.3}$$

Symmetry infinitesimals of the two-dimensional wave equation (15) in circular spatial coordinates are

$$\begin{aligned}
\xi_r &= a_3\cos\theta + a_4\sin\theta + a_5 r + a_6 t\sin\theta + a_7 t\cos\theta + a_8 r t\\
&\qquad + a_9\frac{(r^2+t^2)}{2}\sin\theta + a_{10}\frac{(r^2+t^2)}{2}\cos\theta\\
\xi_\theta &= a_1 - a_3\frac{\sin\theta}{r} + a_4\frac{\cos\theta}{r} + a_6\frac{t}{r}\cos\theta - a_7\frac{t}{r}\sin\theta\\
&\qquad + a_9\left(\frac{t^2}{r} - r\right)\frac{\cos\theta}{2} + a_{10}\left(r - \frac{t^2}{r}\right)\frac{\sin\theta}{2}\\
\xi_t &= a_2 + a_5 t + a_6 r\sin\theta + a_7 r\cos\theta + a_8\frac{(r^2+t^2)}{2} + a_9 r t\sin\theta + a_{10} r t\cos\theta\\
\eta_u &= -a_8\frac{ut}{2} - a_9\frac{ur}{2}\sin\theta - a_{10}\frac{ur}{2}\cos\theta + a_{11} u.
\end{aligned} \tag{A.4}$$

Symmetry infinitesimals of the PDE (17) reduced from the two-dimensional wave equation are

$$\begin{aligned}\xi_R &= b_2(1-R^2)\sin\theta + b_3(1-R^2)\cos\theta + b_5\sqrt{1-R^2}\sin\theta \\ &\quad + b_6\sqrt{1-R}\cos\theta + b_7 R\sqrt{1-R^2} \\ \xi_\theta &= b_1 + b_2\frac{\cos\theta}{R} - b_3\frac{\sin\theta}{R} + b_5\frac{\sqrt{1-R^2}\cos\theta}{R} - b_6\frac{\sqrt{1-R^2}\sin\theta}{R} \\ \eta_w &= b_4 w\end{aligned} \tag{A.5}$$

where the symmetries associated with the constants $b_1 \rightarrow b_4$ are inherited and those with $b_5 \rightarrow b_7$ are Type-II hidden symmetries. Symmetry infinitesimals of the ODE (20) reduced from the reduced PDE of the two-dimensional wave equation are

$$\begin{aligned}\xi_R &= R\sqrt{1-R^2}\left\{c_1 + c_2\ln\left(\frac{R}{1+\sqrt{1-R^2}}\right) + c_3 w + c_4 w\ln\left(\frac{R}{1+\sqrt{1-R^2}}\right)\right. \\ &\quad \left. + c_5\left[\ln\left(\frac{R}{1+\sqrt{1-R^2}}\right)\right]^2\right\} \\ \eta_w &= c_4 w^2 + c_5\ln\left(\frac{R}{1+\sqrt{1-R^2}}\right) + c_6 + c_7\ln\left(\frac{R}{1+\sqrt{1-R^2}}\right) + c_8 w.\end{aligned} \tag{A.6}$$

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